

# DERIVATIVES CHARACTERIZATION OF BERGMAN-ORLICZ SPACES AND APPLICATIONS

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**ABSTRACT.** It is well known that a function is in a Bergman space of the unit ball if and only if it satisfies some Hardy-type inequalities. We extend this fact to Bergman-Orlicz spaces. As applications, we obtain Gustavsson-Peetre interpolation of two Bergman-Orlicz spaces and we completely characterize symbols of bounded or compact Cesàro-type operators on Bergman-Orlicz spaces, extending known results for classical weighted Bergman spaces.

## 1. INTRODUCTION

Let  $z = (z_1, \dots, z_n)$  and  $w = (w_1, \dots, w_n)$  be vectors in  $\mathbb{C}^n$ . We write

$$\langle z, w \rangle = z_1 \overline{w_1} + \dots + z_n \overline{w_n}$$

and  $|z|^2 = \langle z, z \rangle = |z_1|^2 + \dots + |z_n|^2$ .

Given a function  $\Phi : [0, \infty) \rightarrow [0, \infty)$ , We say  $\Phi$  is a growth function if it is a continuous and non-decreasing function.

We denote by  $d\nu$  the Lebesgue measure on  $\mathbb{B}^n$  the unit ball of  $\mathbb{C}^n$ , and  $d\sigma$  the normalized measure on  $\mathbb{S}^n = \partial\mathbb{B}^n$  the boundary of  $\mathbb{B}^n$ . As usual, we denote by  $\mathcal{H}(\mathbb{B}^n)$  the space of holomorphic functions on  $\mathbb{B}^n$ .

For  $\alpha > -1$ , we write  $d\nu_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha d\nu(z)$ , where  $c_\alpha$  is taken such that  $\nu_\alpha(\mathbb{B}^n) = 1$ .

For  $\Phi$  a growth function, the Orlicz space  $L_\alpha^\Phi(\mathbb{B}^n)$  is the space of functions  $f$  such that

$$\|f\|_{\alpha, \Phi} := \int_{\mathbb{B}^n} \Phi(|f(z)|) d\nu_\alpha(z) < \infty.$$

The weighted Bergman-Orlicz space  $\mathcal{A}_\alpha^\Phi(\mathbb{B}^n)$  is the subspace of  $L_\alpha^\Phi(\mathbb{B}^n)$  consisting of holomorphic functions.

We define on  $\mathcal{A}_\alpha^\Phi(\mathbb{B}^n)$  the following (quasi)-norm

$$(1) \quad \|f\|_{\alpha, \Phi}^{lux} := \inf\{\lambda > 0 : \int_{\mathbb{B}^n} \Phi\left(\frac{|f(z)|}{\lambda}\right) d\nu_\alpha(z) \leq 1\}$$

which is finite for  $f \in \mathcal{A}_\alpha^\Phi(\mathbb{B}^n)$  (see [21]).

We observe that for  $\Phi(t) = t^p$ , the corresponding Bergman-Orlicz space is the classical weighted Bergman spaces denoted by  $\mathcal{A}_\alpha^p(\mathbb{B}^n)$  and defined by

$$\|f\|_{p, \alpha}^p = \|f\|_{\mathcal{A}_\alpha^p}^p := \int_{\mathbb{B}^n} |f(z)|^p d\nu_\alpha(z) < \infty.$$

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Recall that two growth functions  $\Phi_1$  and  $\Phi_2$  are said equivalent if there exists some constant  $c$  such that

$$\frac{1}{c}\Phi_1\left(\frac{t}{c}\right) \leq \Phi_2(t) \leq c\Phi_1(ct)$$

and observe that two equivalent growth functions define the same Orlicz space.

We recall that given an analytic function  $f$  on  $\mathbb{B}^n$ , the radial derivative  $\mathcal{R}f$  of  $f$  is defined by

$$\mathcal{R}f(z) = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z).$$

We recall also that the gradient of  $f \in H(\mathbb{B}^n)$  is defined by

$$\nabla f(z) = \left( \frac{\partial f}{\partial z_1}(z), \dots, \frac{\partial f}{\partial z_n}(z) \right).$$

The invariant gradient at  $z$  of the analytic function  $f$  is defined by

$$\tilde{\nabla}f(z) = \nabla(f \circ \phi_z)(0)$$

where  $\phi_z$  is the automorphism of  $\mathbb{B}^n$  mapping 0 to  $z$ .

We have the following inequalities between the above derivatives (see [29, Lemma 2.14]):

$$(2) \quad (1 - |z|^2)|\mathcal{R}f(z)| \leq (1 - |z|^2)|\nabla f(z)| \leq |\tilde{\nabla}f(z)|, \text{ for all } z \in \mathbb{B}^n.$$

The following derivatives characterization of classical weighted Bergman spaces is a well known fact (see [29, Theorem 2.16]).

**THEOREM 1.1.** *Suppose  $\alpha > -1$ ,  $p > 0$ , and  $f$  is holomorphic in  $\mathbb{B}^n$ . Then the following conditions are equivalent.*

- (a)  $f \in \mathcal{A}_\alpha^p(\mathbb{B}^n)$ .
- (b)  $|\tilde{\nabla}f(z)| \in L^p(\mathbb{B}^n, d\nu_\alpha)$
- (c)  $(1 - |z|^2)|\nabla f(z)| \in L^p(\mathbb{B}^n, d\nu_\alpha)$ .
- (d)  $(1 - |z|^2)|\mathcal{R}f(z)| \in L^p(\mathbb{B}^n, d\nu_\alpha)$ .

Our main aim in this note is to extend the above result to Bergman-Orlicz spaces. Let us recall some more definitions.

We say that a growth function  $\Phi$  is of upper type  $q \geq 1$  if there exists  $C > 0$  such that, for  $s > 0$  and  $t \geq 1$ ,

$$(3) \quad \Phi(st) \leq Ct^q\Phi(s).$$

We denote by  $\mathcal{U}^q$  the set of growth functions  $\Phi$  of upper type  $q$ , (for some  $q \geq 1$ ), such that the function  $t \mapsto \frac{\Phi(t)}{t}$  is non-decreasing.

We say that  $\Phi$  is of lower type  $p > 0$  if there exists  $C > 0$  such that, for  $s > 0$  and  $0 < t \leq 1$ ,

$$(4) \quad \Phi(st) \leq Ct^p\Phi(s).$$

We denote by  $\mathcal{L}_p$  the set of growth functions  $\Phi$  of lower type  $p$ , (for some  $p \leq 1$ ), such that the function  $t \mapsto \frac{\Phi(t)}{t}$  is non-increasing.

We also observe that we may always suppose that any  $\Phi \in \mathcal{L}_p$  (resp.  $\mathcal{U}_q$ ), is concave (resp. convex) and that  $\Phi$  is a  $\mathcal{C}^1$  function with derivative  $\Phi'(t) \asymp \frac{\Phi(t)}{t}$ .

Our main result is the following.

**THEOREM 1.2.** *Suppose  $\alpha > -1$ . Assume that  $\Phi \in \mathcal{U}^q \cup \mathcal{L}_p$ , and  $f$  is holomorphic in  $\mathbb{B}^n$ . Then the following conditions are equivalent.*

- (a)  $f \in \mathcal{A}_\alpha^\Phi(\mathbb{B}^n)$ .
- (b)  $|\tilde{\nabla} f(z)| \in L^\Phi(\mathbb{B}^n, d\nu_\alpha)$
- (c)  $(1 - |z|^2)|\nabla f(z)| \in L^\Phi(\mathbb{B}^n, d\nu_\alpha)$ .
- (d)  $(1 - |z|^2)|\mathcal{R}f(z)| \in L^\Phi(\mathbb{B}^n, d\nu_\alpha)$ .

As applications, we characterize the Gustavsson-Peetre interpolate of two Bergman-Orlicz spaces and symbols of bounded Cesàro-type operators on Bergman-Orlicz spaces.

## 2. PRELIMINARY RESULTS

We give in this section some useful tools needed in our presentation.

**2.1. Some properties of growth functions.** We recall that the complementary function  $\Psi$  of the convex growth function  $\Phi$ , is the function defined from  $\mathbb{R}_+$  onto itself by

$$(5) \quad \Psi(s) = \sup_{t \in \mathbb{R}_+} \{ts - \Phi(t)\}.$$

We observe that if  $\Phi \in \mathcal{U}^q$ , then  $\Psi$  is a growth function of lower type such that the function which  $t \mapsto \frac{\Psi(t)}{t}$  is non-decreasing.

We say that  $\Phi$  satisfies the  $\Delta_2$ -condition if there exists a constant  $K > 1$  such that, for any  $t \geq 0$ ,

$$(6) \quad \Phi(2t) \leq K\Phi(t).$$

We say that the growth function  $\Phi$  satisfies the  $\nabla_2$ -condition whenever both  $\Phi$  and its complementary satisfy the  $\Delta_2$ -condition.

For  $\Phi$  a  $\mathcal{C}^1$  growth function, the lower and the upper indices of  $\Phi$  are respectively defined by

$$a_\Phi := \inf_{t>0} \frac{t\Phi'(t)}{\Phi(t)} \quad \text{and} \quad b_\Phi := \sup_{t>0} \frac{t\Phi'(t)}{\Phi(t)}.$$

We recall that when  $\Phi$  is convex, then  $1 \leq a_\Phi \leq b_\Phi < \infty$  and, if  $\Phi$  is concave, then  $0 < a_\Phi \leq b_\Phi \leq 1$ . We observe with [7, Lemma 2.6] that a convex growth function satisfies the  $\nabla_2$ -condition if and only if  $1 < a_\Phi \leq b_\Phi < \infty$ .

It is easy to see that if  $\Phi$  is a  $\mathcal{C}^1$  growth function. Then the functions  $\frac{\Phi(t)}{t^{a_\Phi}}$  and  $\frac{\Phi^{-1}(t)}{t^{b_\Phi}}$  are increasing. As a consequence, we have the following useful fact.

**LEMMA 2.1.** *Let  $\Phi \in \mathcal{L}_p$ . Then the growth function  $\Phi_p$ , defined by  $\Phi_p(t) = \Phi(t^{1/p})$  is in  $\mathcal{U}^q$  for some  $q \geq 1$ .*

We also make the following observation.

**PROPOSITION 2.2.** *The following assertion holds:*

$$\Phi \in \mathcal{L}_p \text{ if and only if } \Phi^{-1} \in \mathcal{U}^{1/p}.$$

We observe that if  $\Phi$  is of upper type (resp. lower type)  $p_1$ , then it is of upper type (resp. lower type)  $p_2$  for any  $\infty > p_2 > p_1$  (resp.  $p_2 < p_1 < \infty$ ). Hence, when we say  $\Phi \in \mathcal{U}^q$  (resp.  $\Phi \in \mathcal{L}_p$ ), we suppose that  $q$  (resp.  $p$ ) is the smallest (resp. biggest) number  $q_1$  (resp.  $p_1$ ) such that  $\Phi$  is of upper type  $q_1$  (resp. lower type  $p_1$ ). We also observe that  $a_\Phi$  (resp.  $b_\Phi$ ) coincides with the biggest (resp. smallest) number  $p$  such that  $\Phi$  is of lower (resp. upper) type  $p$ .

## 2.2. Operators on Orlicz spaces.

**DEFINITION 2.3.** *Let  $\Phi$  be a growth function. A linear operator  $T$  defined on  $L^\Phi(\mathbb{B}^n, d\nu_\alpha)$  is said to be of mean strong type  $(\Phi, \Phi)_\alpha$  if*

$$(7) \quad \int_{\mathbb{B}^n} \Phi(|Tf|) d\nu_\alpha(z) \leq C \int_{\mathbb{B}^n} \Phi(|f|) d\nu_\alpha(z)$$

for any  $f \in L^\Phi(\mathbb{B}^n, d\nu_\alpha)$ , and  $T$  is said to be mean weak type  $(\Phi, \Phi)_\alpha$  if

$$(8) \quad \sup_{t>0} \Phi(t) \nu_\alpha(\{z \in \mathbb{B}^n : |Tf(z)| > t\}) \leq C \int_{\mathbb{B}^n} \Phi(|f|) d\nu_\alpha(z)$$

for any  $f \in L^\Phi(\mathbb{B}^n, d\nu_\alpha)$ , where  $C$  is independent of  $f$ .

We observe that the mean strong type  $(t^p, t^p)_\alpha$  is the usual strong type  $(p, p)$  coincide. We also note if the operator  $T$  is of mean strong type  $(\Phi, \Phi)_\alpha$ , then  $T$  is bounded on  $L^\Phi(\mathbb{B}^n, d\nu_\alpha)$ .

The following result is adapted from [7, Theorem 4.3].

**THEOREM 2.4.** *Let  $\Phi_0, \Phi_1$  and  $\Phi_2$  be three convex growth functions. Suppose that their upper and lower indices satisfy the following condition*

$$(9) \quad 1 \leq a_{\Phi_0} \leq b_{\Phi_0} < a_{\Phi_2} \leq b_{\Phi_2} < a_{\Phi_1} \leq b_{\Phi_1} < \infty.$$

*If  $T$  is of mean weak types  $(\Phi_0, \Phi_0)_\alpha$  and  $(\Phi_1, \Phi_1)_\alpha$ , then it is of mean strong type  $(\Phi_2, \Phi_2)_\alpha$ .*

Let  $\beta > -1$  be and consider the operator  $P_\beta$  defined for functions  $f$  on  $\mathbb{B}^n$  by

$$(10) \quad P_\beta(f)(z) = \int_{\mathbb{B}^n} \frac{f(\xi)}{(1 - \langle z, \xi \rangle)^{n+1+\beta}} d\nu_\beta(\xi).$$

The operator  $P_\beta$  is the Bergman projection, that is the orthogonal projection of  $L^2(\mathbb{B}^n, d\nu_\beta)$  onto its closed subspace  $\mathcal{A}_\beta^2(\mathbb{B}^n)$ . We have the following result.

**THEOREM 2.5.** *Let  $\alpha, \beta > -1$ . Let  $\Phi$  be a convex growth function and denote by  $a_\Phi$  its lower indice. Assume that there is  $1 < p_0 < a_\Phi$  such that  $\alpha + 1 < p_0(\beta + 1)$ . Then  $P_\beta$  is of mean strong type  $(\Phi, \Phi)_\alpha$ .*

*Proof.* This result is well known when  $\Phi$  is a power function (see for example [29, Theorem 2.10]). It follows in particular that  $P_\beta$  is bounded on  $L^{p_0}(\mathbb{B}^n, d\nu_\alpha)$  and on  $L^{p_1}(\mathbb{B}^n, d\nu_\alpha)$  for  $p_1 > b_\Phi$ . Hence from the interpolation result Theorem 2.4 we deduce that  $P_\beta$  is of mean strong type  $(\Phi, \Phi)_\alpha$ .  $\square$

In particular, we have the following.

**THEOREM 2.6.** *Let  $\alpha > -1$ . Assume that  $\Phi \in \mathcal{U}^q$  and satisfies the  $\nabla_2$ -condition. Then the Bergman projection  $P_\alpha$  is bounded on  $L^\Phi(\mathbb{B}^n, d\nu_\alpha)$ .*

**2.3. Some useful estimates and test functions.** The next proposition gives pointwise estimates for functions in  $\mathcal{A}_\alpha^\Phi(\mathbb{B}^n)$ ,  $\Phi \in \mathcal{L}_p \cup \mathcal{U}^q$  (see [21, 24])

**LEMMA 2.7.** *Let  $\Phi \in \mathcal{L}_p \cup \mathcal{U}^q$  and  $\alpha > -1$ . There is a constant  $C > 1$  such that for any  $f \in \mathcal{A}_\alpha^\Phi(\mathbb{B}^n)$ ,*

$$(11) \quad |f(z)| \leq C\Phi^{-1}\left(\frac{1}{(1-|z|^2)^{n+1+\alpha}}\right) \|f\|_{\alpha, \Phi}^{lux}.$$

We refer to [28, Lemma 2.15] for the following result.

**LEMMA 2.8.** *Let  $0 < p \leq 1$ . Then there is a constant  $C > 0$  such that for any  $f \in \mathcal{A}_\alpha^p(\mathbb{B}^n)$ ,*

$$(12) \quad \int_{\mathbb{B}^n} |f(z)|(1-|z|^2)^{(\frac{1}{p}-1)(n+1+\alpha)} d\nu_\alpha(z) \leq C\|f\|_{\alpha, p}^p.$$

The following gives example of functions in Bergman-Orlicz spaces. We refer to [21, 24] for a proof.

**LEMMA 2.9.** *Let  $-1 < \alpha < \infty$ ,  $a \in \mathbb{B}^n$ . Let  $k > 1$ . Suppose that  $\Phi \in \mathcal{L}_p \cup \mathcal{U}^q$ . Then the following function is in  $\mathcal{A}_\alpha^\Phi(\mathbb{B}^n)$*

$$f_a(z) = \Phi^{-1}\left(\frac{1}{(1-|a|)^{n+1+\alpha}}\right) \left(\frac{1-|a|^2}{1-\langle z, a \rangle}\right)^{k(n+1+\alpha)}.$$

Moreover,  $\|f_a\|_{\mathcal{A}_\alpha^\Phi}^{lux} \lesssim 1$ .

### 3. PROOF OF THEOREM 1.2

Let us start with the following result.

**LEMMA 3.1.** *Let  $\alpha > -1$ . Assume that  $\Phi \in \mathcal{U}^q \cup \mathcal{L}_p$ . Then there exists a constant  $C > 0$  such that for any  $f \in \mathcal{A}_\alpha^\Phi(\mathbb{B}^n)$ ,*

$$(13) \quad \int_{\mathbb{B}^n} \Phi(|\widetilde{\nabla} f(z)|) d\nu_\alpha(z) \leq C \int_{\mathbb{B}^n} \Phi(|f(z) - f(0)|) d\nu_\alpha(z).$$

*Proof.* We follow the proof of [29, Theorem 2.16] making some crucial modifications where needed. We start by recalling that if  $\Phi \in \mathcal{U}^q$ , then  $\mathcal{A}_\alpha^\Phi(\mathbb{B}^n)$  continuously embeds into  $\mathcal{A}_\alpha^1(\mathbb{B}^n)$ , and  $\mathcal{A}_\alpha^\Phi(\mathbb{B}^n)$  continuously embeds into  $\mathcal{A}_\alpha^p(\mathbb{B}^n)$  when  $\Phi \in \mathcal{L}_p$ . Let  $\beta > \alpha$ . Put

$$(14) \quad p_\Phi = \begin{cases} 1 & \text{if } \Phi \in \mathcal{U}^q \\ p & \text{if } \Phi \in \mathcal{L}_p. \end{cases}$$

It follows from [29, Lemma 2.4] that there exists  $C_1 > 0$  such that for any  $g \in H(\mathbb{B}^n)$ ,

$$|\nabla g(0)|^{p_\Phi} \leq C_1 \int_{\mathbb{B}^n} |g(w)|^{p_\Phi} d\nu_\beta(w).$$

Put  $g = f \circ \phi_z$ ,  $z \in \mathbb{B}^n$ , where  $\phi_z$  is the automorphism of  $\mathbb{B}^n$  such that  $\phi_z(0) = z$ . We obtain

$$|\widetilde{\nabla} f(z)|^{p_\Phi} \leq C_1 \int_{\mathbb{B}^n} |f(w)|^{p_\Phi} \frac{(1-|z|^2)^{n+1+\beta}}{|1-\langle z, w \rangle|^{2(n+1+\beta)}} d\nu_\beta(w).$$

We observe with the help of [20, Proposition 1.4.10] that  $\frac{(1-|z|^2)^{n+1+\beta}}{|1-\langle z, w \rangle|^{2(n+1+\beta)}} d\nu_\beta(w)$  is up to a constant a probability measure. It follows using the convexity of

$$\Phi_p(t) = \begin{cases} \Phi(t) & \text{if } \Phi \in \mathcal{U}^q \\ \Phi(t^{\frac{1}{p}}) & \text{if } \Phi \in \mathcal{L}_p \end{cases}$$

and Jensen's Inequality that

$$\Phi(|\widetilde{\nabla} f(z)|) \leq C_2 \int_{\mathbb{B}^n} \Phi(|f(w)|) \frac{(1-|z|^2)^{n+1+\beta}}{|1-\langle z, w \rangle|^{2(n+1+\beta)}} d\nu_\beta(w).$$

Finally, integrating both sides of the last inequality over  $\mathbb{B}^n$  with respect to  $d\nu_\alpha(z)$  and applying Fubini's Theorem and [20, Proposition 1.4.10], we obtain

$$\int_{\mathbb{B}^n} \Phi(|\widetilde{\nabla} f(z)|) d\nu_\alpha \leq C_2 \int_{\mathbb{B}^n} \Phi(|f(z)|) d\nu_\alpha(z).$$

Replacing  $f$  by  $f - f(0)$ , we have

$$\int_{\mathbb{B}^n} \Phi(|\widetilde{\nabla} f(z)|) d\nu_\alpha \leq C_2 \int_{\mathbb{B}^n} \Phi(|f(z) - f(0)|) d\nu_\alpha(z).$$

The proof is complete.  $\square$

We also obtain the following.

**LEMMA 3.2.** *Let  $\alpha > -1$ . Assume that  $\Phi \in \mathcal{U}^q$  or  $\Phi \in \mathcal{L}_p$ . Then there exists a constant  $C > 0$  such that for any  $f \in H(\mathbb{B}^n)$  such that  $(1 - |z|^2)\mathcal{R}f(z) \in L^\Phi(\mathbb{B}^n, d\nu_\alpha)$ ,*

$$(15) \quad \int_{\mathbb{B}^n} \Phi(|f(z) - f(0)|) d\nu_\alpha(z) \leq C \int_{\mathbb{B}^n} \Phi((1 - |z|^2)|\mathcal{R}f(z)|) d\nu_\alpha(z).$$

*Proof.* Let  $f \in H(\mathbb{B}^n)$  be such that  $(1 - |z|^2)\mathcal{R}f(z) \in L^\Phi(\mathbb{B}^n, d\nu_\alpha)$ . Then following the proof of [29, Theorem 2.16] at page 51, we have that for  $\beta$  large enough,

$$(16) \quad |f(z) - f(0)| \leq \int_{\mathbb{B}^n} \frac{|\mathcal{R}f(w)|}{|1 - \langle z, w \rangle|^{n+\beta}} d\nu_\beta(w).$$

Let us first consider the case of  $\Phi \in \mathcal{U}^q$ . Fix  $p$  so that  $1 < p < a_\Phi$ , and observe that (16) is equivalent to

$$|f(z) - f(0)| \leq CP_{\beta-1}((1 - |\cdot|^2)|\mathcal{R}f(\cdot)|)(z).$$

Taking  $\beta$  large enough so that

$$0 < \alpha + 1 < p\beta,$$

we obtain from Theorem 2.5 that

$$\int_{\mathbb{B}^n} \Phi(|f(z) - f(0)|) d\nu_\alpha(z) \leq C \int_{\mathbb{B}^n} \Phi((1 - |z|^2)|\mathcal{R}f(z)|) d\nu_\alpha(z).$$

We next consider the case of  $\Phi \in \mathcal{L}_p$ . We assume that  $\beta$  is large enough so that

$$\beta = \frac{n+1+\gamma}{p} - (n+1), \quad \gamma > \alpha + p.$$

Rewriting (16) as

$$|f(z) - f(0)| \leq \int_{\mathbb{B}^n} \left| \frac{\mathcal{R}f(w)}{(1 - \langle z, w \rangle)^{n+\beta}} \right| (1 - |w|^2)^{(\frac{1}{p}-1)(n+1+\gamma)} d\nu(w),$$

we obtain from Lemma 2.8 that

$$|f(z) - f(0)|^p \leq C \int_{\mathbb{B}^n} \left| \frac{\mathcal{R}f(w)}{(1 - \langle z, w \rangle)^{n+\beta}} \right|^p d\nu_\gamma(w),$$

or equivalently,

$$\begin{aligned} |f(z) - f(0)|^p &\leq C \int_{\mathbb{B}^n} \frac{((1 - |w|^2)|\mathcal{R}f(w)|)^p}{|1 - \langle z, w \rangle|^{n+1+(\gamma-p)}} d\nu_{\gamma-p}(w) \\ &= CP_{\gamma-p}([(1 - |\cdot|^2)|\mathcal{R}f(\cdot)|]^p)(z). \end{aligned}$$

As the growth function  $t \mapsto \Phi_p(t) = \Phi(t^{\frac{1}{p}})$  is in  $\mathcal{U}^q$ , proceeding as in the first part of this proof, we obtain

$$\begin{aligned} \int_{\mathbb{B}^n} \Phi(|f(z) - f(0)|) d\nu_\alpha(z) &\leq C \int_{\mathbb{B}^n} \Phi_p([(1 - |\cdot|^2)|\mathcal{R}f(\cdot)|]^p)(z) d\nu_\alpha(z) \\ &= C \int_{\mathbb{B}^n} \Phi((1 - |z|^2)|\mathcal{R}f(z)|) d\nu_\alpha(w). \end{aligned}$$

The proof is complete. □

We can now prove Theorem 1.2.

*Proof of Theorem 1.2.* That (b) implies (c) and (c) implies (d) follow from (2). That (a) implies (b) is Lemma 3.1 and that (d) implies (a) is Lemma 3.2. The proof is complete. □

#### 4. APPLICATIONS

**4.1. The Gustavsson-Peetre interpolation of two Bergman-Orlicz spaces.** Our aim in this section is to give an application of Theorem 1.2 to a generalized interpolation of quasi-Banach spaces due to J. Gustavsson and J. Peetre. For this, we first introduce several definitions and results.

**DEFINITION 4.1.** A function  $\rho : [0, \infty) \rightarrow [0, \infty)$  is said to be pseudo-concave, if it is continuous on  $(0, \infty)$  and

$$\rho(s) \leq \max(1, \frac{s}{t})\rho(t), \text{ for all } s, t > 0.$$

We denote by  $\mathcal{T}$  the set of all pseudo-concave functions.

**DEFINITION 4.2.** A function  $\rho \in \mathcal{T}$  is said to be in  $\mathcal{T}^{+-}$ , if

$$\sup_x \frac{\rho(\lambda x)}{\rho(x)} = o(\max(1, \lambda)) \text{ as } x \rightarrow 0 \text{ or } x \rightarrow \infty.$$

As example of functions in  $\mathcal{T}^{+-}$  we have of course concave functions. The following function was provided in [10] as a non trivial element in  $\mathcal{T}^{+-}$ :

$$\rho(t) = t^\theta (\log(e + t))^\alpha (e + \frac{1}{t})^\beta$$

where  $0 < \theta < 1$ , and  $\alpha, \beta$  are real numbers.

**DEFINITION 4.3** (J. Gustavsson and J. Peetre [10]). *Let  $A_0$  and  $A_1$  be quasi-Banach spaces, both embedded in a Hausdorff topological space  $\mathcal{A}$ . We call  $\vec{A} = (A_0, A_1)$  a quasi-Banach couple. Let  $\rho \in \mathcal{T}$ . We denote by  $\langle \vec{A} \rangle_\rho = \langle A_0, A_1 \rangle_\rho$  the space of all elements  $a \in \sum(\vec{A}) := A_0 + A_1$  such that there exists a sequence  $u = \{u_\alpha\}_{\alpha \in \mathbb{Z}}$  of elements  $u_\alpha \in \Delta(\vec{A}) := A_0 \cap A_1$  such that*

$$(17) \quad a = \sum_{\alpha \in \mathbb{Z}} u_\alpha \quad (\text{convergence in } \sum(\vec{A})),$$

*for every finite subset  $F \subset \mathbb{Z}$  and every real sequence  $\xi = \{\xi_\alpha\}_{\alpha \in F}$  with  $|\xi_\alpha| \leq 1$ , we have*

$$\left\| \sum_F \frac{\xi_\alpha 2^{k\alpha} u_\alpha}{\rho(2^\alpha)} \right\| \leq C \quad (k = 0, 1)$$

*with  $C$  independent of  $F$  and  $\xi$ .*

*We equip the space  $\langle \vec{A} \rangle_\rho$  with the semi-norm*

$$\|a\|_{\langle \vec{A} \rangle_\rho} := \inf_u C,$$

*where the infimum is taken over all admissible sequences  $u$  as above.*

As observed in [10], if  $\rho \in \mathcal{T}^{+-}$ , then  $\|a\|_{\langle \vec{A} \rangle_\rho}$  is a quasi-norm and  $\langle \vec{A} \rangle_\rho$  is a quasi-Banach space.

Let us recall that if  $T : A \rightarrow B$  is a continuous linear operator between two quasi-Banach spaces  $A$  and  $B$ , the operator norm of  $T$  denoted  $\|T\|_{A \rightarrow B}$  is defined by

$$\|T\|_{A \rightarrow B} := \sup_{x \in A, x \neq 0} \frac{\|Tx\|_B}{\|x\|_A}.$$

**PROPOSITION 4.4.** ([10, Proposition 6.1.]) *Let  $\vec{A} = (A_0, A_1)$  and  $\vec{B} = (B_0, B_1)$  be two quasi-Banach couples. If  $T : \vec{A} \rightarrow \vec{B}$  is a continuous linear mapping, that is the restriction  $T_i = T|_{A_i} : A_i \rightarrow B_i$ , ( $i = 0, 1$ ) is continuous, then  $T : \langle \vec{A} \rangle_\rho \rightarrow \langle \vec{B} \rangle_\rho$  is continuous and*

$$\|T\|_{\langle \vec{A} \rangle_\rho \rightarrow \langle \vec{B} \rangle_\rho} \leq \max(\|T\|_{A_0 \rightarrow B_0}, \|T\|_{A_1 \rightarrow B_1}).$$

*That is the functor  $(A_0, A_1) \mapsto \langle \vec{A} \rangle_\rho$  is an interpolation space.*

We call  $\langle \vec{A} \rangle_\rho$  the Gustavsson-Peetre interpolate of  $A_0$  and  $A_1$ . As observed in [10], when  $\rho(s) = s^\theta$ ,  $0 < \theta < 1$ ,  $\langle \vec{A} \rangle_\rho$  corresponds to the complex interpolation (see also [5, 14, 18]). For more on complex interpolation, we refer to the book [2].

The following is a restriction of [10, Theorem 7.3] to the class of growth functions we are interested in and our spaces.

**PROPOSITION 4.5.** *Let  $\Phi_i \in \mathcal{U}^q$ ,  $i = 0, 1$  satisfying the  $\nabla_2$ -condition, and let  $\alpha > -1$ . Assume that  $\rho \in \mathcal{T}^{+-}$  and let  $\Phi$  be defined by*

$$(18) \quad \Phi^{-1} = \Phi_0^{-1} \rho\left(\frac{\Phi_1^{-1}}{\Phi_0^{-1}}\right).$$

*Then*

$$L^\Phi(\mathbb{B}^n, d\nu_\alpha) = L^{\vec{\Phi}}_\rho(\mathbb{B}^n, d\nu_\alpha) = \langle L^{\Phi_0}(\mathbb{B}^n, d\nu_\alpha), L^{\Phi_1}(\mathbb{B}^n, d\nu_\alpha) \rangle_\rho$$



with equivalence of (quasi)-norms. In particular,  $\vec{L}^\Phi_\rho(\mathbb{B}^n, d\nu_\alpha)$  is an interpolation space with respect to  $(L^{\Phi_0}(\mathbb{B}^n, d\nu_\alpha), L^{\Phi_1}(\mathbb{B}^n, d\nu_\alpha))$ .

It is easy to check using the definition of a pseudo-concave function, that given  $\Phi_0, \Phi_1 \in \mathcal{U}^q$ , the growth function  $\Phi$  defined by (18) is of upper type  $q > 1$ .

We can now state our main result of this section.

**THEOREM 4.6.** *Let  $\Phi_i \in \mathcal{U}^q$ ,  $i = 0, 1$  satisfying the  $\nabla_2$ -condition, and let  $\alpha > -1$ . Assume that  $\rho \in \mathcal{T}^{+-}$  and let  $\Phi$  be defined as in (18). Then*

$$\mathcal{A}_\alpha^\Phi(\mathbb{B}^n) = \langle \mathcal{A}_\alpha^{\Phi_0}(\mathbb{B}^n), \mathcal{A}_\alpha^{\Phi_1}(\mathbb{B}^n) \rangle_\rho$$

with equivalence of (quasi)-norms.

*Proof.* As  $\Phi_0$  and  $\Phi_1$  satisfy the  $\nabla_2$ -condition, we have from Theorem 2.6 that the Bergman projection  $P_\alpha$  maps  $L^{\Phi_0}(\mathbb{B}^n, d\nu_\alpha)$  boundedly onto  $\mathcal{A}_\alpha^{\Phi_0}(\mathbb{B}^n)$ , and it maps  $L^{\Phi_1}(\mathbb{B}^n, d\nu_\alpha)$  boundedly onto  $\mathcal{A}_\alpha^{\Phi_1}(\mathbb{B}^n)$ . It follows from Proposition 4.4 that  $P_\alpha$  maps  $L^\Phi(\mathbb{B}^n, d\nu_\alpha) = \langle L^{\Phi_0}(\mathbb{B}^n, d\nu_\alpha), L^{\Phi_1}(\mathbb{B}^n, d\nu_\alpha) \rangle_\rho$  boundedly into  $\langle \mathcal{A}_\alpha^{\Phi_0}(\mathbb{B}^n), \mathcal{A}_\alpha^{\Phi_1}(\mathbb{B}^n) \rangle_\rho$ . As  $P_\alpha(L^\Phi(\mathbb{B}^n, d\nu_\alpha)) = \mathcal{A}_\alpha^\Phi(\mathbb{B}^n)$ , we conclude that

$$(19) \quad \mathcal{A}_\alpha^\Phi(\mathbb{B}^n) \subset \langle \mathcal{A}_\alpha^{\Phi_0}(\mathbb{B}^n), \mathcal{A}_\alpha^{\Phi_1}(\mathbb{B}^n) \rangle_\rho.$$

Conversely, if we denote by  $L$  the operator defined on  $H(\mathbb{B}^n)$  by

$$L(f)(z) := (1 - |z|^2)\mathcal{R}f(z), \quad z \in \mathbb{B}^n,$$

then following Theorem 1.2 and specially Lemma 3.2, we have that  $L$  maps  $\mathcal{A}_\alpha^{\Phi_0}(\mathbb{B}^n)$  boundedly into  $L^{\Phi_0}(\mathbb{B}^n, d\nu_\alpha)$ , and it maps  $\mathcal{A}_\alpha^{\Phi_1}(\mathbb{B}^n)$  into  $L^{\Phi_1}(\mathbb{B}^n, d\nu_\alpha)$ . It follows once more from Proposition 4.4 that  $L$  maps  $\langle \mathcal{A}_\alpha^{\Phi_0}(\mathbb{B}^n), \mathcal{A}_\alpha^{\Phi_1}(\mathbb{B}^n) \rangle_\rho$  into  $L^\Phi(\mathbb{B}^n, d\nu_\alpha)$ . That is if  $f \in \langle \mathcal{A}_\alpha^{\Phi_0}(\mathbb{B}^n), \mathcal{A}_\alpha^{\Phi_1}(\mathbb{B}^n) \rangle_\rho$ , then the function  $z \mapsto (1 - |z|^2)\mathcal{R}f(z)$  belongs to  $L^\Phi(\mathbb{B}^n, d\nu_\alpha)$ , which by Theorem 1.2 is equivalent to saying that  $f \in \mathcal{A}_\alpha^\Phi(\mathbb{B}^n)$ . We deduce that

$$(20) \quad \langle \mathcal{A}_\alpha^{\Phi_0}(\mathbb{B}^n), \mathcal{A}_\alpha^{\Phi_1}(\mathbb{B}^n) \rangle_\rho \subset \mathcal{A}_\alpha^\Phi(\mathbb{B}^n).$$

From (19) and (20), we conclude that

$$\langle \mathcal{A}_\alpha^{\Phi_0}(\mathbb{B}^n), \mathcal{A}_\alpha^{\Phi_1}(\mathbb{B}^n) \rangle_\rho = \mathcal{A}_\alpha^\Phi(\mathbb{B}^n).$$

The proof is complete.  $\square$

Restricting to power functions, we deduce the following.

**COROLLARY 4.7.** *Let  $1 \leq p_0 < p_1 < \infty$ , and let  $\alpha > -1$ . Assume that  $\rho \in \mathcal{T}^{+-}$  and let  $\Phi$  be defined by  $\Phi^{-1}(t) = t^{\frac{1}{p_0}} \rho(t^{\frac{1}{p_1} - \frac{1}{p_0}})$ . Then*

$$\mathcal{A}_\alpha^\Phi(\mathbb{B}^n) = \langle \mathcal{A}_\alpha^{p_0}(\mathbb{B}^n), \mathcal{A}_\alpha^{p_1}(\mathbb{B}^n) \rangle_\rho$$

with equivalence of (quasi)-norms.

Note that the above corollary tells us that given two classical Bergman spaces with the same weight, their Gustavsson-Peetre interpolation space is in general a Bergman-Orlicz space while the complex interpolation of weighted Bergman spaces always gives another weighted Bergman space (see [29]).

**4.2. Boundedness and compactness of weighted Cesàro-type integrals.** For  $g \in \mathcal{H}(\mathbb{B}^n)$  with  $g(0) = 0$ , we consider the following integral-type operator defined on  $\mathcal{H}(\mathbb{B}^n)$  by

$$T_g f(z) = \int_0^1 f(tz) \mathcal{R}g(tz) \frac{dt}{t}.$$

The operator  $T_g$  is the so-called extended Cesàro operator introduced in [11]. The boundedness and compactness of  $T_g$  on the weighted Bergman space  $\mathcal{A}_\alpha^p(\mathbb{B}^n)$  were studied by J. Xiao [27]. In [16] the same questions between different weighted Bergman spaces were studied. Note that Z. Hu also considered the boundedness and compactness of  $T_g$  between weighted Bergman spaces for a large class of weights [13].

We aim in this section to characterize symbols  $g$  such that  $T_g$  is a bounded or compact operator from a weighted Bergman-Orlicz space to itself.

We first prove an estimate for derivative of functions in Bergman-Orlicz spaces.

**LEMMA 4.8.** *Let  $\Phi \in \mathcal{L}_p \cup \mathcal{U}^q$  and  $\alpha > -1$ . Then there are two positive constants  $C_1$  and  $C_2$  such that for any  $f \in \mathcal{A}_\alpha^\Phi(\mathbb{B}^n)$ ,*

$$(21) \quad |\nabla f(z)| \leq \frac{C_1}{1 - |z|^2} \Phi^{-1} \left( \frac{C_2}{(1 - |z|^2)^{n+1+\alpha}} \|f\|_{\alpha, \Phi}^{lux} \right), \text{ for any } z \in \mathbb{B}^n.$$

*Proof.* Let us start by considering the case where  $\Phi \in \mathcal{U}^q$ . We observe that in this case,  $\mathcal{A}_\alpha^\Phi(\mathbb{B}^n)$  continuously embeds into  $\mathcal{A}_\alpha^1(\mathbb{B}^n)$ . Hence, for any  $f \in \mathcal{A}_\alpha^\Phi(\mathbb{B}^n)$ , and any  $z \in \mathbb{B}^n$ ,

$$f(z) = \int_{\mathbb{B}^n} \frac{f(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w).$$

Thus for any  $j = 1, \dots, n$ ,

$$\frac{\partial f}{\partial z_j}(z) = c \int_{\mathbb{B}^n} \frac{\overline{w}_j f(w)}{(1 - \langle z, w \rangle)^{n+2+\alpha}} d\nu_\alpha(w).$$

It follows easily that

$$\frac{1 - |z|^2}{\|f\|_{\alpha, \Phi}^{lux}} \left| \frac{\partial f}{\partial z_j}(z) \right| \leq C \int_{\mathbb{B}^n} \frac{|f(w)|}{\|f\|_{\alpha, \Phi}^{lux}} \frac{1 - |z|^2}{(1 - \langle z, w \rangle)^{n+2+\alpha}} d\nu_\alpha(w).$$

It is easy to see using [20, Proposition 1.4.10] that  $\frac{1 - |z|^2}{(1 - \langle z, w \rangle)^{n+2+\alpha}} d\nu_\alpha(w)$  is up to a constant a probability measure. Hence using the convexity of  $\Phi$  and Jensen's Inequality, we obtain

$$\begin{aligned} \Phi \left( \frac{1 - |z|^2}{\|f\|_{\alpha, \Phi}^{lux}} \left| \frac{\partial f}{\partial z_j}(z) \right| \right) &\leq C \int_{\mathbb{B}^n} \Phi \left( \frac{|f(w)|}{\|f\|_{\alpha, \Phi}^{lux}} \right) \frac{1 - |z|^2}{(1 - \langle z, w \rangle)^{n+2+\alpha}} d\nu_\alpha(w) \\ &\leq \frac{C}{(1 - |z|^2)^{n+1+\alpha}} \int_{\mathbb{B}^n} \Phi \left( \frac{|f(w)|}{\|f\|_{\alpha, \Phi}^{lux}} \right) d\nu_\alpha(w) \\ &\leq \frac{C}{(1 - |z|^2)^{n+1+\alpha}}. \end{aligned}$$

Hence

$$\left| \frac{\partial f}{\partial z_j}(z) \right| \leq \frac{1}{1-|z|^2} \Phi^{-1} \left( \frac{C}{(1-|z|^2)^{n+1+\alpha}} \right) \|f\|_{\alpha, \Phi}^{lux}, \text{ for any } z \in \mathbb{B}^n$$

from which follows (21).

We now consider the case where  $\Phi \in \mathcal{L}_p$ . We recall that in this case  $\Phi$  is of lower type  $0 < p \leq 1$ . Let  $\beta > -1$  be large enough (this will be more precise in the next lines). As above, we have that

$$(22) \quad \left| \frac{\partial f}{\partial z_j}(z) \right| \leq C \int_{\mathbb{B}^n} \frac{|f(w)|}{|1-\langle z, w \rangle|^{n+2+\beta}} d\nu_\beta(w).$$

We assume that  $\beta = \frac{n+1+\gamma}{p} - (n+1)$  with  $\gamma > \alpha + p$ . Then using Lemma 2.8, we obtain from (22) that

$$(23) \quad \left| \frac{\partial f}{\partial z_j}(z) \right|^p \leq C \int_{\mathbb{B}^n} \left| \frac{f(w)}{(1-\langle z, w \rangle)^{n+2+\beta}} \right|^p d\nu_\gamma(w)$$

or equivalently,

$$(24) \quad \left| \frac{1-|z|^2}{\|f\|_{\alpha, \Phi}^{lux}} \frac{\partial f}{\partial z_j}(z) \right|^p \leq C \int_{\mathbb{B}^n} \left| \frac{f(w)}{\|f\|_{\alpha, \Phi}^{lux}} \right|^p \frac{(1-|z|^2)^p}{|1-\langle z, w \rangle|^{(n+2+\beta)p}} d\nu_\gamma(w).$$

One easily checks that  $\frac{(1-|z|^2)^p}{|1-\langle z, w \rangle|^{(n+2+\beta)p}} d\nu_\gamma(w)$  is up to a constant, a probability measure. Hence using that the function  $\Phi_p : t \mapsto \Phi(t) = \Phi(t^{\frac{1}{p}})$  is convex and Jensen's Inequality, we obtain that

$$\Phi_p \left( \left| \frac{1-|z|^2}{\|f\|_{\alpha, \Phi}^{lux}} \frac{\partial f}{\partial z_j}(z) \right|^p \right) \leq C \int_{\mathbb{B}^n} \Phi_p \left( \left| \frac{f(w)}{\|f\|_{\alpha, \Phi}^{lux}} \right|^p \right) \frac{(1-|z|^2)^p}{|1-\langle z, w \rangle|^{(n+2+\beta)p}} d\nu_\gamma(w).$$

Hence

$$\begin{aligned} \Phi \left( \left| \frac{1-|z|^2}{\|f\|_{\alpha, \Phi}^{lux}} \frac{\partial f}{\partial z_j}(z) \right| \right) &\leq C \int_{\mathbb{B}^n} \Phi \left( \left| \frac{f(w)}{\|f\|_{\alpha, \Phi}^{lux}} \right| \right) \frac{(1-|z|^2)^p}{|1-\langle z, w \rangle|^{(n+2+\beta)p}} d\nu_\gamma(w) \\ &= C \int_{\mathbb{B}^n} \Phi \left( \left| \frac{f(w)}{\|f\|_{\alpha, \Phi}^{lux}} \right| \right) \frac{(1-|z|^2)^p}{|1-\langle z, w \rangle|^{n+1+\gamma+p}} d\nu_\gamma(w) \\ &\leq \frac{C}{(1-|z|^2)^{n+1+\alpha}} \int_{\mathbb{B}^n} \Phi \left( \left| \frac{f(w)}{\|f\|_{\alpha, \Phi}^{lux}} \right| \right) d\nu_\alpha(w) \\ &\leq \frac{C}{(1-|z|^2)^{n+1+\alpha}}. \end{aligned}$$

That is

$$\left| \frac{\partial f}{\partial z_j}(z) \right| \leq \frac{1}{1-|z|^2} \Phi^{-1} \left( \frac{C}{(1-|z|^2)^{n+1+\alpha}} \right) \|f\|_{\alpha, \Phi}^{lux}, \text{ for any } z \in \mathbb{B}^n$$

from which follows (21). The proof is complete.  $\square$

We can now prove the following.

**THEOREM 4.9.** *Let  $\Phi \in \mathcal{L}_p \cup \mathcal{U}^q$  and  $\alpha > -1$ . Assume  $g$  is a holomorphic function on  $\mathbb{B}^n$  with  $g(0) = 0$ . Then the operator  $T_g$  is bounded on  $\mathcal{A}_\alpha^\Phi(\mathbb{B}^n)$  if and only if*

$$(25) \quad M := \sup_{z \in \mathbb{B}^n} (1 - |z|^2) |\mathcal{R}g(z)| < \infty.$$

Moreover, if we denote by  $\|T_g\|$  the operator norm of  $T_g$ , then

$$\|T_g\| \sim M.$$

*Proof.* Let us first assume that (25) holds. Then

$$\begin{aligned} I &:= \int_{\mathbb{B}^n} \Phi \left( \frac{(1 - |z|^2) |\mathcal{R}T_g f(z)|}{M \|f\|_{\Phi, \alpha}^{lux}} \right) d\nu_\alpha(z) \\ &= \int_{\mathbb{B}^n} \Phi \left( \frac{(1 - |z|^2) |\mathcal{R}g(z)| |f(z)|}{M \|f\|_{\Phi, \alpha}^{lux}} \right) d\nu_\alpha(z) \\ &\leq \int_{\mathbb{B}^n} \Phi \left( \frac{|f(z)|}{\|f\|_{\Phi, \alpha}^{lux}} \right) d\nu_\alpha(z) \leq 1. \end{aligned}$$

Hence from Theorem 1.2, we deduce that  $T_g f \in \mathcal{A}_\alpha^\Phi(\mathbb{B}^n)$  for any  $f \in \mathcal{A}_\alpha^\Phi(\mathbb{B}^n)$ . Moreover, from Lemma 3.2, we deduce that

$$\|T_g f\|_{\Phi, \alpha}^{lux} \leq M \|f\|_{\Phi, \alpha}^{lux}$$

for any  $f \in \mathcal{A}_\alpha^\Phi(\mathbb{B}^n)$ . It follows that

$$\|T_g\| \leq M.$$

Conversely, let us assume that for  $g \in H(\mathbb{B}^n)$  with  $g(0) = 0$ ,  $T_g$  is bounded on  $\mathcal{A}_\alpha^\Phi(\mathbb{B}^n)$ . Then from Lemma 4.8 we have that there is a constant  $C > 0$  such that for any  $f \in \mathcal{A}_\alpha^\Phi(\mathbb{B}^n)$  and any  $z \in \mathbb{B}^n$ ,

$$(1 - |z|^2) |\mathcal{R}T_g f(z)| \leq C \Phi^{-1} \left( \frac{1}{(1 - |z|^2)^{n+1+\alpha}} \right) \|T_g f\|_{\Phi, \alpha}^{lux}$$

which leads to

$$(26) \quad (1 - |z|^2) |\mathcal{R}g(z)| |f(z)| \leq C \Phi^{-1} \left( \frac{1}{(1 - |z|^2)^{n+1+\alpha}} \right) \|T_g\| \|f\|_{\Phi, \alpha}^{lux}.$$

Let  $a \in \mathbb{B}^n$  be fixed and consider the function  $f_a$  defined on  $\mathbb{B}^n$  by

$$f_a(z) = \Phi^{-1} \left( \frac{1}{(1 - |a|^2)^{n+1+\alpha}} \right) \left( \frac{1 - |a|^2}{1 - \langle z, a \rangle} \right)^{2(n+1+\alpha)}.$$

We recall with Lemma 2.9 that  $f_a \in \mathcal{A}_\alpha^\Phi(\mathbb{B}^n)$  and  $\|f_a\|_{\Phi, \alpha}^{lux} \lesssim 1$ . Let us test (26) with the function  $f = f_a$ , for  $a$  fixed. We obtain that for any  $z \in \mathbb{B}^n$ ,

$$\begin{aligned} &(1 - |z|^2) |\mathcal{R}g(z)| \Phi^{-1} \left( \frac{1}{(1 - |a|^2)^{n+1+\alpha}} \right) \left| \frac{1 - |a|^2}{1 - \langle z, a \rangle} \right|^{2(n+1+\alpha)} \\ &\leq C \Phi^{-1} \left( \frac{1}{(1 - |z|^2)^{n+1+\alpha}} \right) \|T_g\|. \end{aligned}$$

Putting  $z = a$ , we obtain that

$$(1 - |a|^2) |\mathcal{R}g(a)| \leq C \|T_g\|.$$

As  $a$  was taken arbitrary in  $\mathbb{B}^n$ , we deduce that there is a constant  $C > 0$  such

$$M := \sup_{a \in \mathbb{B}^n} (1 - |a|^2) |\mathcal{R}g(a)| \leq C \|T_g\|.$$

The proof is complete.  $\square$

Recall that the spaces of all holomorphic functions satisfying (25) is called the Bloch space. The equivalent characterizations in Theorem 1.2 show that the Bloch space embeds continuously into  $\mathcal{A}_\alpha^\Phi(\mathbb{B}^n)$  for any  $\Phi \in \mathcal{U}^q \cup \mathcal{L}_p$ .

We next consider compactness of the operators  $T_g$ . For this, we need the following compactness criteria which can be proved following the usual arguments (see [6]).

**LEMMA 4.10.** *Let  $\Phi \in \mathcal{U}^q \cup \mathcal{L}_p$ , and let  $\alpha > -1$ . Let  $g \in H(\mathbb{B}^n)$  with  $g(0) = 0$ . Suppose that  $T_g : \mathcal{A}_\alpha^\Phi(\mathbb{B}^n) \rightarrow \mathcal{A}_\alpha^\Phi(\mathbb{B}^n)$  is bounded, then*

$$T_g : \mathcal{A}_\alpha^{\Phi_p}(\mathbb{B}^n) \rightarrow \mathcal{A}_\alpha^\Phi(\mathbb{B}^n)$$

*is compact if and only if for every sequence  $(f_j)$  in the unit ball of  $\mathcal{A}_\alpha^\Phi(\mathbb{B}^n)$  which converges to 0 uniformly on compact subsets of  $\mathbb{B}^n$ , one has*

$$\|T_g(f_j)\|_{\Phi, \alpha}^{lux} \rightarrow 0 \text{ as } j \rightarrow \infty.$$

We have the following result.

**THEOREM 4.11.** *Let  $\Phi \in \mathcal{L}_p \cup \mathcal{U}^q$  and  $\alpha > -1$ . Assume  $g$  is a holomorphic function on  $\mathbb{B}^n$ . Then the operator  $T_g : \mathcal{A}_\alpha^\Phi(\mathbb{B}^n) \rightarrow \mathcal{A}_\alpha^\Phi(\mathbb{B}^n)$  is compact if and only if*

$$(27) \quad \lim_{|z| \rightarrow 1} (1 - |z|^2) |\mathcal{R}g(z)| = 0.$$

*Proof.* Let us first assume that  $T_g$  is compact. Let  $\{a_j\}_{j \in \mathbb{N}}$  be a sequence in  $\mathbb{B}^n$  such that  $\lim_{j \rightarrow \infty} |a_j| = 1$ . Consider the sequence of holomorphic functions on  $\mathbb{B}^n$  given by

$$f_j(z) = \Phi^{-1} \left( \frac{1}{(1 - |a_j|^2)^{n+1+\alpha}} \right) \left( \frac{1 - |a_j|^2}{1 - \langle z, a_j \rangle} \right)^{k(n+1+\alpha)}$$

with  $k > 1$  to be precised where needed. From Lemma 2.9, we know that the sequence  $\{f_j\}_{j \in \mathbb{N}}$  is uniformly bounded in  $\mathcal{A}_\alpha^\Phi(\mathbb{B}^n)$ . Also, we have that if  $\Phi \in \mathcal{U}^q$ , then as  $\Phi^{-1}$  is concave and as  $k > 1$ ,

$$|f_j(z)| \leq \frac{(1 - |a_j|^2)^{(k-1)(n+1+\alpha)}}{|1 - \langle z, a_j \rangle|^{k(n+1+\alpha)}} \rightarrow 0$$

as  $j \rightarrow \infty$  on compact subsets of  $\mathbb{B}^n$ . If  $\Phi \in \mathcal{L}_p$ , then as  $\Phi^{-1} \in \mathcal{U}^{\frac{1}{p}}$ , taking  $k > \frac{1}{p}$ , we obtain using (3) that

$$|f_j(z)| \leq C \frac{(1 - |a_j|^2)^{(k-\frac{1}{p})(n+1+\alpha)}}{|1 - \langle z, a_j \rangle|^{k(n+1+\alpha)}} \rightarrow 0$$

as  $j \rightarrow \infty$  on compact subsets of  $\mathbb{B}^n$ .

Using Lemma 4.8, we obtain that there is a constant  $C > 0$  such that each  $j \in \mathbb{N}$ , and for any  $z \in \mathbb{B}^n$ ,

$$(1 - |z|^2)|\mathcal{R}T_g f_j(z)| \leq C\Phi^{-1}\left(\frac{1}{(1 - |z|^2)^{n+1+\alpha}}\right) \|T_g(f_j)\|_{\Phi,\alpha}^{lux}$$

or equivalently,

$$\begin{aligned} & (1 - |z|^2)\Phi^{-1}\left(\frac{1}{(1 - |a_j|^2)^{n+1+\alpha}}\right) \left|\frac{1 - |a_j|^2}{1 - \langle z, a_j \rangle}\right|^{k(n+1+\alpha)} |\mathcal{R}g(z)| \\ & \leq C\Phi^{-1}\left(\frac{1}{(1 - |z|^2)^{n+1+\alpha}}\right) \|T_g(f_j)\|_{\Phi,\alpha}^{lux}. \end{aligned}$$

Putting in particular  $z = a_j$ , we obtain

$$(1 - |a_j|^2)|\mathcal{R}g(a_j)| \leq C\|T_g(f_j)\|_{\Phi,\alpha}^{lux}.$$

As  $\|T_g(f_j)\|_{\Phi,\alpha}^{lux} \rightarrow 0$  as  $j \rightarrow \infty$ , we deduce that

$$\lim_{j \rightarrow \infty} (1 - |a_j|^2)|\mathcal{R}g(a_j)| = 0$$

which leads to

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)|\mathcal{R}g(z)| = 0.$$

Conversely, let us assume that the holomorphic function  $g$  satisfies (27). Note that this implies that  $g \in \mathcal{A}_\alpha^\Phi(\mathbb{B}^n)$  and that for any  $\varepsilon > 0$ , there exists  $\eta$  such that

$$(28) \quad (1 - |z|^2)|\mathcal{R}g(z)| < \varepsilon$$

for any  $z \in \mathbb{B}^n$  such that  $\eta < |z| < 1$ .

Let us start by proving that  $T_g$  is bounded on  $\mathcal{A}_\alpha^\Phi(\mathbb{B}^n)$ . Let

$$K = \max\{1, C\Phi^{-1}\left(\frac{1}{(1 - \eta^2)^{n+1+\alpha}}\right), \|g\|_{\Phi,\alpha}^{lux}\}$$

where  $C$  is (11). We have at first that for any  $f \in \mathcal{A}_\alpha^\Phi(\mathbb{B}^n)$ ,

$$\begin{aligned} L &:= \int_{\mathbb{B}^n} \Phi\left(\frac{(1 - |z|^2)|\mathcal{R}T_g f(z)|}{K^2\|f\|_{\Phi,\alpha}^{lux}}\right) d\nu_\alpha(z) \\ &= \int_{\mathbb{B}^n} \Phi\left(\frac{(1 - |z|^2)|\mathcal{R}g(z)||f(z)|}{K^2\|f\|_{\Phi,\alpha}^{lux}}\right) d\nu_\alpha(z) \\ &\leq \int_{|z| \leq \eta} \Phi\left(\frac{(1 - |z|^2)|\mathcal{R}g(z)||f(z)|}{K^2\|f\|_{\Phi,\alpha}^{lux}}\right) d\nu_\alpha(z) + \\ &\quad \int_{|z| > \eta} \Phi\left(\frac{(1 - |z|^2)|\mathcal{R}g(z)||f(z)|}{K^2\|f\|_{\Phi,\alpha}^{lux}}\right) d\nu_\alpha(z). \end{aligned}$$

Using the pointwise estimate (11) and the definition of the constant  $K$ , we obtain

$$\begin{aligned}
L_1 &:= \int_{|z| \leq \eta} \Phi \left( \frac{(1 - |z|^2) |\mathcal{R}g(z)| |f(z)|}{K^2 \|f\|_{\Phi, \alpha}^{lux}} \right) d\nu_\alpha(z) \\
&\leq \int_{|z| \leq \eta} \Phi \left( \frac{(1 - |z|^2) |\mathcal{R}g(z)| C \Phi^{-1} \left( \frac{1}{(1 - |z|^2)^{n+1+\alpha}} \right)}{K^2} \right) d\nu_\alpha(z) \\
&\leq \int_{|z| \leq \eta} \Phi \left( \frac{(1 - |z|^2) |\mathcal{R}g(z)| C \Phi^{-1} \left( \frac{1}{(1 - \eta^2)^{n+1+\alpha}} \right)}{K^2} \right) d\nu_\alpha(z).
\end{aligned}$$

It follows from the equivalent characterization in Theorem 1.2 that

$$\begin{aligned}
L_1 &\leq \int_{|z| \leq \eta} \Phi \left( \frac{(1 - |z|^2) |\mathcal{R}g(z)|}{\|g\|_{\Phi, \alpha}^{lux}} \right) d\nu_\alpha(z) \\
&\leq \int_{\mathbb{B}^n} \Phi \left( \frac{(1 - |z|^2) |\mathcal{R}g(z)|}{\|g\|_{\Phi, \alpha}^{lux}} \right) d\nu_\alpha(z) \\
&\lesssim \int_{\mathbb{B}^n} \Phi \left( \frac{|g(z)|}{\|g\|_{\Phi, \alpha}^{lux}} \right) d\nu_\alpha(z) \leq 1.
\end{aligned}$$

That is

$$(29) \quad \int_{|z| \leq \eta} \Phi \left( \frac{(1 - |z|^2) |\mathcal{R}g(z)| |f(z)|}{K^2 \|f\|_{\Phi, \alpha}^{lux}} \right) d\nu_\alpha(z) \lesssim 1.$$

Using the estimate (28), we obtain

$$\begin{aligned}
L_2 &:= \int_{|z| > \eta} \Phi \left( \frac{(1 - |z|^2) |\mathcal{R}g(z)| |f(z)|}{K^2 \|f\|_{\Phi, \alpha}^{lux}} \right) d\nu_\alpha(z) \\
&\leq \int_{|z| > \eta} \Phi \left( \frac{\varepsilon |f(z)|}{K^2 \|f\|_{\Phi, \alpha}^{lux}} \right) d\nu_\alpha(z) \\
&\leq \int_{\mathbb{B}^n} \Phi \left( \frac{|f(z)|}{\|f\|_{\Phi, \alpha}^{lux}} \right) d\nu_\alpha(z) \\
&\leq 1.
\end{aligned}$$

That is

$$(30) \quad \int_{|z| > \eta} \Phi \left( \frac{(1 - |z|^2) |\mathcal{R}g(z)| |f(z)|}{K^2 \|f\|_{\Phi, \alpha}^{lux}} \right) d\nu_\alpha(z) \leq 1.$$

From (29), (30) and the equivalent characterizations in Theorem 1.2, we deduce that  $T_g$  is bounded on  $\mathcal{A}_\alpha^\Phi(\mathbb{B}^n)$ .

Now, let  $\{f_j\}_{j \in \mathbb{N}}$  be a sequence in the unit ball of  $\mathcal{A}_\alpha^\Phi(\mathbb{B}^n)$  which converges to 0 uniformly on compact subsets of  $\mathbb{B}^n$ . Then there exists an integer  $j_0 > 0$  such that for any  $j > j_0$ ,

$$\sup_{0 < |z| \leq \eta} |f_j(z)| < \varepsilon.$$

Let  $M := \max\{1, \|g\|_{\Phi, \alpha}^{lux}\}$ . Then we obtain for any  $j > j_0$ ,

$$\begin{aligned} L &:= \int_{\mathbb{B}^n} \Phi \left( \frac{(1 - |z|^2) |\mathcal{R}T_g f_j(z)|}{M} \right) d\nu_\alpha(z) \\ &= \int_{\mathbb{B}^n} \Phi \left( \frac{(1 - |z|^2) |\mathcal{R}g(z)| |f_j(z)|}{M} \right) d\nu_\alpha(z) \\ &\leq \int_{|z| \leq \eta} \Phi \left( \frac{(1 - |z|^2) |\mathcal{R}g(z)| |f_j(z)|}{M} \right) d\nu_\alpha(z) + \\ &\quad \int_{|z| > \eta} \Phi \left( \frac{(1 - |z|^2) |\mathcal{R}g(z)| |f_j(z)|}{M} \right) d\nu_\alpha(z). \end{aligned}$$

Using the convexity of  $\Phi$  if  $\Phi \in \mathcal{U}^q$  and condition (4) if  $\Phi \in \mathcal{L}_p$  and the definition of  $p_\Phi$  in (14), it follows that

$$\begin{aligned} L &\lesssim \varepsilon^{p_\Phi} \int_{|z| \leq \eta} \Phi \left( \frac{(1 - |z|^2) |\mathcal{R}g(z)|}{M} \right) d\nu_\alpha(z) + \varepsilon^{p_\Phi} \int_{|z| > \eta} \Phi \left( \frac{|f_j(z)|}{M} \right) d\nu_\alpha(z) \\ &\lesssim \varepsilon^{p_\Phi} \int_{\mathbb{B}^n} \Phi \left( \frac{|g(z)|}{M} \right) d\nu_\alpha(z) + \varepsilon^{p_\Phi} \int_{\mathbb{B}^n} \Phi \left( \frac{|f_j(z)|}{M} \right) d\nu_\alpha(z) \\ &\leq 2\varepsilon^{p_\Phi}. \end{aligned}$$

It follows that  $\int_{\mathbb{B}^n} \Phi \left( \frac{(1 - |z|^2) |\mathcal{R}T_g f_j(z)|}{M} \right) d\nu_\alpha(z) \rightarrow 0$  as  $j \rightarrow \infty$ . Hence that  $\int_{\mathbb{B}^n} \Phi \left( \frac{|T_g f_j(z)|}{M} \right) d\nu_\alpha(z) \rightarrow 0$  as  $j \rightarrow \infty$ . This implies that  $\|T_g f_j\|_{\Phi, \alpha}^{lux} \rightarrow 0$  as  $j \rightarrow \infty$ . Thus  $T_g$  is a compact operator on  $\mathcal{A}_\alpha^\Phi(\mathbb{B}^n)$ . The proof is complete.  $\square$

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